**Math 131 B, Lecture 2**
**Analysis**

**Sample Midterm 2**

**Instructions:** You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

**Problem 1**. *10pts.*

Match each general statement on the first list below with its consequence on the real line on the second list.

List One: General Theorems

- (A) If $S \subset M$ is a dense subset of a metric space $M$, any continuous function $f : M \to T$ is determined by its values on $S$.
- (B) If $S$ is compact, any infinite subset $K$ of $S$ has a limit point in $S$.
- (C) If $p$ is a limit point of a set $S$, every neighbourhood of $p$ contains infinitely many points of $S$.
- (D) The image of a connected set under a continuous map is connected.
- (E) The image of a compact set under a continuous map is compact.

List Two: Consequences in $\mathbb{R}$.

- (1) Let $f : [a, b] \to \mathbb{R}$ be continuous. If $x = \sup\{f(x) : x \in [a, b]\}$ and $y = \inf\{f(x) : x \in [a, b]\}$. Then there is $c, d \in [a, b]$ such that $f(c) = x$ and $f(d) = y$.
- (2) Every bounded sequence of real numbers has a convergent subsequence.
- (3) Let $f : [a, b] \to \mathbb{R}$ be continuous. If $x$ is a real number which is between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $f(c) = x$.
- (4) If two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ agree on all rational numbers, then they are the same function.
- (5) A real number $y$ is a subsequential limit of a sequence $\{x_n\}$ if and only if $\{n : x_n \in (y - \epsilon, y + \epsilon)\}$ is infinite for all $\epsilon > 0$.

**Problem 2**.

(a) [5pts.] Give a definition of a connected metric space $M$.

(b) [5pts.] Let $S \subset M$ be a subset of a metric space. Prove or disprove: If $\overline{S}$ is connected, $S$ is connected.

**Problem 3.**

(a) [5pts.] Let $\{f_n\}$ be a sequence of functions $f_n : S \to T$. What does it mean for $f_n$ to converge uniformly to a function $f : S \to T$?

(b) [5pts.] Prove that if $f_n \to f$ uniformly and each $f_n$ is integrable, $f$ is integrable.

**Problem 4.**

(a) [5pts.] State the Weierstrass M-test.

(b) [5pts.] Prove that the function $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n(1+n^2x^2)}$ is continuous. What is its antiderivative?

**Problem 5**.

(a) [5pts.] Define the Cauchy product of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$.

(b) [5pts.] Let $c_n = \sum_{k=1}^{n} \frac{k}{n+1-k}$. What function is represented by the power series $\sum_{n=0}^{\infty} c_n x^n$ on $(-1, 1)$?